異なる n 個の整数 1, 2, 3, $\cdots$, n の中から重複を許して 2 個の整数を選び，すべての組合せについて．2 数の和および積をたし合わせたものをそれぞれ $S(n)$, $T(n)$ とする．$n \geqq 2$ であるとき，次の問いに答えよ．

(1) $S(3)$, $T(3)$ を求めよ．
(2) $S(n)$, $T(n)$ を n の式で表せ．

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【答】

(1) $S(3) = 24,\ T(3) = 25$
(2) $S(n) = \dfrac{1}{2}n(n+1)^2,\ T(n) = \dfrac{1}{24}n(n+1)(n+2)(3n+1)$

【解答】

(1) $S(n)$, $T(n)$ の定義より

$$\begin{aligned} S(3) &= (1+1)+(1+2)+(1+3)+(2+2)+(2+3)+(3+3) \\ &= (2+3+4)+(4+5)+6 \\ &= \mathbf{24} \end{aligned} \qquad \cdots\cdots(\text{答})$$

$$\begin{aligned} T(3) &= 1\cdot 1+1\cdot 2+1\cdot 3+2\cdot 2+2\cdot 3+3\cdot 3 \\ &= (1+2+3)+(4+6)+9 \\ &= \mathbf{25} \end{aligned} \qquad \cdots\cdots(\text{答})$$

である．

(2) (1) と同じように計算すると

$$\begin{aligned} S(n) = (1+1)&+(1+2)+(1+3)+\cdots+(1+n) \\ &+(2+2)+(2+3)+\cdots+(2+n) \\ &\qquad\quad +(3+3)+\cdots+(3+n) \\ &\qquad\qquad\qquad\qquad\qquad\vdots \\ &\qquad\qquad\qquad\qquad\quad +(n+n) \end{aligned}$$

$$\begin{aligned} &= \sum_{k=1}^{n}\sum_{j=k}^{n}(k+j) \qquad (\text{行の和を加える}) \\ &= \sum_{k=1}^{n}\frac{(n-k+1)(2k+k+n)}{2} \qquad \left(\because\ \frac{(\text{項数})(\text{初項}+\text{末項})}{2}\right) \\ &= \frac{1}{2}\sum_{k=1}^{n}\{(-k+n+1)(3k+n)\} \\ &= \frac{1}{2}\sum_{k=1}^{n}\{-3k^2+(2n+3)k+n(n+1)\} \\ &= \frac{1}{2}\left\{-3\cdot\frac{1}{6}n(n+1)(2n+1)+(2n+3)\cdot\frac{1}{2}n(n+1)+n(n+1)\cdot n\right\} \\ &= \frac{1}{2}\cdot\frac{1}{2}n(n+1)\{-(2n+1)+(2n+3)+2n\} \\ &= \frac{1}{4}n(n+1)(2n+2) \\ &= \boldsymbol{\frac{1}{2}n(n+1)^2} \end{aligned} \qquad \cdots\cdots(\text{答})$$

である．

- 列の和を求め加えていくと

$$
\begin{aligned}
S(n) &= \sum_{k=1}^{n}\sum_{j=1}^{k}(j+k) \\
&= \sum_{k=1}^{n}\frac{k(1+k+2k)}{2} \qquad \left(\because\ \frac{(\text{項数})(\text{初項}+\text{末項})}{2}\right) \\
&= \sum_{k=1}^{n}\left(\frac{3}{2}k^2+\frac{1}{2}k\right) \\
&= \frac{3}{2}\cdot\frac{1}{6}n(n+1)(2n+1)+\frac{1}{2}\cdot\frac{1}{2}n(n+1) \\
&= \frac{1}{4}n(n+1)\{(2n+1)+1\} \\
&= \frac{1}{2}n(n+1)^2
\end{aligned}
$$

- 次のように考えてもよい.

$$
\begin{aligned}
2S(n) = \{&(1+1)+(1+2)+(1+3)+\cdots+(1+n) \\
&+(2+1)+(2+2)+(2+3)+\cdots+(2+n) \\
&+(3+1)+(3+2)+(3+3)+\cdots+(3+n) \\
&\qquad\vdots \\
&+(n+1)+(n+2)+(n+3)+\cdots+(n+n)\} \\
&+\{(1+1)+(2+2)+(3+3)+\cdots+(n+n)\} \\
=\ &\{n\cdot 1+(1+2+3+\cdots+n)\}+\{n\cdot 2+(1+2+3+\cdots+n)\} \\
&+\{n\cdot 3+(1+2+3+\cdots+n)\}+\cdots+\{n\cdot n+(1+2+3+\cdots+n)\} \\
&+2(1+2+3+\cdots+n) \\
=\ &n(1+2+3+\cdots+n)+n(1+2+3+\cdots+n)+2(1+2+3+\cdots+n) \\
=\ &(2n+2)(1+2+3+\cdots+n) \\
=\ &(2n+2)\cdot\frac{1}{2}n(n+1) \\
=\ &n(n+1)^2
\end{aligned}
$$

$$\therefore\quad S(n)=\frac{1}{2}n(n+1)^2$$

また

$$
\begin{aligned}
T(n) &= (1\cdot 1)+(1\cdot 2)+(1\cdot 3)+\cdots+(1\cdot n)\\
&\qquad +(2\cdot 2)+(2\cdot 3)+\cdots+(2\cdot n)\\
&\qquad\qquad +(3\cdot 3)+\cdots+(3\cdot n)\\
&\qquad\qquad\qquad\qquad\vdots\\
&\qquad\qquad\qquad\qquad +(n\cdot n)\\
&= \sum_{k=1}^{n}\sum_{j=k}^{n}(kj) \qquad (\text{行の和を加える})\\
&= \sum_{k=1}^{n}\left\{k\cdot\frac{(n-k+1)(k+n)}{2}\right\} \quad \left(\because\ \frac{(\text{項数})(\text{初項}+\text{末項})}{2}\right)\\
&= \frac{1}{2}\sum_{k=1}^{n}\{-k^3+k^2+n(n+1)k\}\\
&= \frac{1}{2}\left\{-\frac{1}{4}n^2(n+1)^2+\frac{1}{6}n(n+1)(2n+1)+n(n+1)\cdot\frac{1}{2}n(n+1)\right\}\\
&= \frac{1}{2}\cdot\frac{1}{12}n(n+1)\{-3n(n+1)+2(2n+1)+6n(n+1)\}\\
&= \frac{1}{24}n(n+1)(3n^2+7n+2)\\
&= \boldsymbol{\frac{1}{24}n(n+1)(n+2)(3n+1)} \qquad \cdots\cdots(\text{答})
\end{aligned}
$$

である.

- 列の和を求め加えていくと

$$
\begin{aligned}
T(n) &= \sum_{k=1}^{n}\sum_{j=1}^{k}(jk)\\
&= \sum_{k=1}^{n}\left\{\frac{k(k+1)}{2}\cdot k\right\}\\
&= \frac{1}{2}\left\{\frac{1}{4}n^4(n+1)^2+\frac{1}{6}n(n+1)(2n+1)\right\}\\
&= \frac{1}{2}\cdot\frac{1}{12}n(n+1)\{3n(n+1)+2(2n+1)\}\\
&= \frac{1}{24}n(n+1)(3n^2+7n+2)\\
&= \frac{1}{24}n(n+1)(n+2)(3n+1)
\end{aligned}
$$

- 次のように考えてもよい.

$$
\begin{aligned}
2T(n) &= \{(1\cdot 1)+(1\cdot 2)+(1\cdot 3)+\cdots+(1\cdot n)\\
&\qquad +(2\cdot 1)+(2\cdot 2)+(2\cdot 3)+\cdots+(2\cdot n)\\
&\qquad +(3\cdot 1)+(3\cdot 2)+(3\cdot 3)+\cdots+(3\cdot n)\\
&\qquad\qquad\vdots\\
&\qquad +(n\cdot 1)+(n\cdot 2)+(n\cdot 3)+\cdots+(n\cdot n)\}\\
&\quad +\{(1\cdot 1)+(2\cdot 2)+(3\cdot 3)+\cdots+(n\cdot n)\}\\
&= (1+2+3+\cdots+n)(1+2+3+\cdots+n)+(1^2+2^2+3^2+\cdots+n^2)\\
&= \frac{1}{4}n^4(n+1)^2+\frac{1}{6}n(n+1)(2n+1)\\
&= \frac{1}{12}n(n+1)\{3n(n+1)+2(2n+1)\}\\
&= \frac{1}{12}n(n+1)(3n^2+7n+2)\\
&= \frac{1}{12}n(n+1)(n+2)(3n+1)
\end{aligned}
$$

$$\therefore\quad T(n)=\frac{1}{24}n(n+1)(n+2)(3n+1)$$