$a,\ b$ を実数とし，$I=\displaystyle\int_0^{\pi}(x+a\sin x+b\cos x)^2\,dx$ とする．次の問いに答えよ．

(1) $I_1=\displaystyle\int_0^{\pi}\sin^2 x\,dx$，$I_2=\displaystyle\int_0^{\pi}\cos x\sin x\,dx$，$I_3=\displaystyle\int_0^{\pi}x\cos x\,dx$ の値をそれぞれ求めよ．

(2) I を $a,\ b$ の式で表せ．

(3) I の最小値，およびそのときの $a,\ b$ の値を求めよ．

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【答】

(1) $I_1=\dfrac{\pi}{2}$，$I_2=0$，$I_3=-2$

(2) $I=\dfrac{\pi}{2}(a^2+b^2)+2\pi a-4b+\dfrac{\pi^3}{3}$

(3) $a=-2,\ b=\dfrac{4}{\pi}$ のとき，最小値 $\dfrac{\pi^4-6\pi^2-24}{3\pi}$

【解答】

(1) 積分を実行していくと

$$I_1=\int_0^{\pi}\sin^2 x\,dx=\int_0^{\pi}\frac{1-\cos 2x}{2}\,dx=\left[\frac{x}{2}-\frac{\sin 2x}{4}\right]_0^{\pi}=\boldsymbol{\frac{\pi}{2}}\qquad\cdots\cdots(\text{答})$$

$$I_2=\int_0^{\pi}\cos x\sin x\,dx=\int_0^{\pi}\frac{\sin 2x}{2}\,dx=\left[-\frac{\cos 2x}{4}\right]_0^{\pi}=\mathbf{0}\qquad\cdots\cdots(\text{答})$$

$$I_3=\int_0^{\pi}x\cos x\,dx=\Big[x\sin x\Big]_0^{\pi}-\int_0^{\pi}1\cdot\sin x\,dx=-\Big[-\cos x\Big]_0^{\pi}=\mathbf{-2}\qquad\cdots\cdots(\text{答})$$

- $I_2=\displaystyle\int_0^{\pi}\cos x\sin x\,dx=\int_0^{\pi}\sin x(\sin x)'\,dx=\left[\frac{\sin^2 x}{2}\right]_0^{\pi}=0$

(2)
$$\begin{aligned}I&=\int_0^{\pi}(x+a\sin x+b\cos x)^2\,dx\\&=\int_0^{\pi}\{x^2+2x(a\sin x+b\cos x)+(a\sin x+b\cos x)^2\}\,dx\\&=\left[\frac{x^3}{3}\right]_0^{\pi}+2a\int_0^{\pi}x\sin x\,dx+2bI_3+\int_0^{\pi}(a^2\sin^2 x+2ab\sin x\cos x+b^2\cos^2 x)\,dx\\&=\frac{\pi^3}{3}+2a\int_0^{\pi}x\sin x\,dx+2b\cdot(-2)+\frac{\pi}{2}a^2+2ab\cdot 0+b^2\int_0^{\pi}\cos^2 x\,dx\end{aligned}$$

ここで

$$\int_0^{\pi}x\sin x\,dx=\Big[x(-\cos x)\Big]_0^{\pi}-\int_0^{\pi}1\cdot(-\cos x)\,dx=\pi+\Big[\sin x\Big]_0^{\pi}=\pi$$

$$\int_0^{\pi}\cos^2 x\,dx=\int_0^{\pi}(1-\sin^2 x)\,dx=\Big[x\Big]_0^{\pi}-I_1=\pi-\frac{\pi}{2}=\frac{\pi}{2}$$

よって

$$
\begin{aligned}
I &= \frac{\pi^3}{3} + 2\pi a - 4b + \frac{\pi}{2}a^2 + \frac{\pi}{2}b^2 \\
&= \boldsymbol{\frac{\pi}{2}(a^2+b^2) + 2\pi a - 4b + \frac{\pi^3}{3}} \qquad \cdots\cdots(\text{答})
\end{aligned}
$$

(3) (2) より

$$
\begin{aligned}
I &= \left(\frac{\pi}{2}a^2 + 2\pi a\right) + \left(\frac{\pi}{2}b^2 - 4b\right) + \frac{\pi^3}{3} \\
&= \frac{\pi}{2}(a+2)^2 - 2\pi + \frac{\pi}{2}\left(b - \frac{4}{\pi}\right)^2 - \frac{8}{\pi} + \frac{\pi^3}{3} \\
&= \frac{\pi}{2}(a+2)^2 + \frac{\pi}{2}\left(b - \frac{4}{\pi}\right)^2 + \frac{\pi^4 - 6\pi^2 - 24}{3\pi}
\end{aligned}
$$

よって，I は

$$
\boldsymbol{a = -2,\ b = \frac{4}{\pi}} \text{ のとき，最小値 } \boldsymbol{\frac{\pi^4 - 6\pi^2 - 24}{3\pi}} \qquad \cdots\cdots(\text{答})
$$

をとる．