

次の極限を求めよ.

$$\lim_{n \rightarrow \infty} \frac{1}{n^2} \left\{ \cos \frac{\pi}{2n} + 2 \cos \frac{2\pi}{2n} + 3 \cos \frac{3\pi}{2n} + \cdots + (n-1) \cos \frac{(n-1)\pi}{2n} \right\}$$

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[答]  $\frac{2}{\pi} - \frac{4}{\pi^2}$

【解答】

区分求積法を用いる.

$$\begin{aligned} & \lim_{n \rightarrow \infty} \frac{1}{n^2} \left\{ \cos \frac{\pi}{2n} + 2 \cos \frac{2\pi}{2n} + 3 \cos \frac{3\pi}{2n} + \cdots + (n-1) \cos \frac{(n-1)\pi}{2n} \right\} \\ &= \lim_{n \rightarrow \infty} \frac{1}{n^2} \sum_{k=1}^{n-1} k \cos \frac{k\pi}{2n} \\ &= \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \frac{k}{n} \cos \frac{k\pi}{2n} \quad \left( \because n \cos \frac{n\pi}{2n} = 0 \right) \\ &= \int_0^1 x \cos \left( \frac{\pi}{2} x \right) dx \\ &= \left[ x \cdot \frac{2}{\pi} \sin \left( \frac{\pi}{2} x \right) \right]_0^1 - \int_0^1 1 \cdot \frac{2}{\pi} \sin \left( \frac{\pi}{2} x \right) dx \\ &= \frac{2}{\pi} - \left[ - \left( \frac{2}{\pi} \right)^2 \cos \left( \frac{\pi}{2} x \right) \right]_0^1 \\ &= \frac{2}{\pi} - \frac{4}{\pi^2} \end{aligned} \qquad \text{.....(答)}$$