

n を正の整数とする.

$$(1) \int_0^{\frac{\pi}{3}} \tan^n \theta d\theta + \int_0^{\frac{\pi}{3}} \tan^{n+2} \theta d\theta を n の式で表せ.$$

$$(2) \int_0^{\frac{\pi}{3}} \tan^7 \theta d\theta を求めよ.$$

(19 千葉大 11)

【答】

$$(1) \frac{(\sqrt{3})^{n+1}}{n+1}$$

$$(2) \frac{15}{4} - \log 2$$

【解答】

(1) 2 つの積分をまとめると

$$\begin{aligned} & \int_0^{\frac{\pi}{3}} \tan^n \theta d\theta + \int_0^{\frac{\pi}{3}} \tan^{n+2} \theta d\theta \\ &= \int_0^{\frac{\pi}{3}} \tan^n \theta (1 + \tan^2 \theta) d\theta \\ &= \int_0^{\frac{\pi}{3}} \tan^n \theta \frac{1}{\cos^2 \theta} d\theta \\ &= \int_0^{\frac{\pi}{3}} \tan^n \theta (\tan \theta)' d\theta \\ &= \left[\frac{\tan^{n+1} \theta}{n+1} \right]_0^{\frac{\pi}{3}} \\ &= \frac{(\sqrt{3})^{n+1}}{n+1} \end{aligned} \quad \dots\dots\text{(答)}$$

$$(2) I_n = \int_0^{\frac{\pi}{3}} \tan^n \theta d\theta おくと, (1) の結果は$$

$$I_n + I_{n+2} = \frac{(\sqrt{3})^{n+1}}{n+1}$$

である. これより

$$\begin{aligned} & \int_0^{\frac{\pi}{3}} \tan^7 \theta d\theta = I_7 \\ &= (I_7 + I_5) - (I_5 + I_3) + (I_3 + I_1) - I_1 \\ &= \frac{(\sqrt{3})^6}{6} - \frac{(\sqrt{3})^4}{4} + \frac{(\sqrt{3})^2}{2} - \int_0^{\frac{\pi}{3}} \frac{\sin \theta}{\cos \theta} d\theta \\ &= \frac{3^3}{6} - \frac{3^2}{4} + \frac{3}{2} + \left[\log |\cos \theta| \right]_0^{\frac{\pi}{3}} \\ &= \frac{9}{2} - \frac{9}{4} + \frac{3}{2} + \log \frac{1}{2} \\ &= \frac{15}{4} - \log 2 \end{aligned} \quad \dots\dots\text{(答)}$$