

$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{dx}{\sin x}$ の値を求めよ.

(19 福岡教大 中教 (数学) 1(3))

【答】 $\log(2 + \sqrt{3}) - \frac{1}{2} \log 3$

【解答】

与式を変形する.

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{dx}{\sin x} = \int_{\pi}^{\frac{\pi}{3}} \frac{\sin x}{\sin^2 x} dx = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sin x}{1 - \cos^2 x} dx$$

$\cos x = t$ とおくと

$$-\sin x dx = dt \quad \begin{array}{c|c} x & \\ \hline \frac{\pi}{6} & \rightarrow \frac{\pi}{3} \\ \frac{\sqrt{3}}{2} & \rightarrow \frac{1}{2} \end{array}$$

であるから

$$\begin{aligned} \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{dx}{\sin x} &= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sin x}{1 - \cos^2 x} dx = \int_{\frac{\sqrt{3}}{2}}^{\frac{1}{2}} \frac{-1}{1 - t^2} dt \\ &= -\frac{1}{2} \int_{\frac{\sqrt{3}}{2}}^{\frac{1}{2}} \left(\frac{1}{1-t} + \frac{1}{1+t} \right) dt \\ &= \frac{1}{2} \left[-\log(1-t) + \log(1+t) \right]_{\frac{\sqrt{3}}{2}}^{\frac{1}{2}} \\ &= \frac{1}{2} \left[\log \frac{1+t}{1-t} \right]_{\frac{\sqrt{3}}{2}}^{\frac{1}{2}} \\ &= \frac{1}{2} \left(\log \frac{2+\sqrt{3}}{2-\sqrt{3}} - \log 3 \right) \\ &= \frac{1}{2} \{ \log(2 + \sqrt{3})^2 - \log 3 \} \\ &= \log(2 + \sqrt{3}) - \frac{1}{2} \log 3 \end{aligned}$$

……(答)