不定積分 $\displaystyle\int\left(2\sqrt{x^2+1}-\frac{1}{\sqrt{x^2+1}}\right)dx$ は，$\boxed{\text{ケ}}$ となる．ただし積分定数を C とする．

(22 産業医大 医 2(6))

【答】

ケ
$x\sqrt{x^2+1}+C$

【解答】

部分積分法を用いると

$$\begin{aligned}\int\sqrt{x^2+1}\,dx &= x\sqrt{x^2+1}-\int x\cdot\frac{2x}{2\sqrt{x^2+1}}\,dx\\ &= x\sqrt{x^2+1}-\int\frac{(x^2+1)-1}{\sqrt{x^2+1}}\,dx\\ &= x\sqrt{x^2+1}-\int\left(\sqrt{x^2+1}-\frac{1}{\sqrt{x^2+1}}\right)dx\end{aligned}$$

よって

$$\int\left(2\sqrt{x^2+1}-\frac{1}{\sqrt{x^2+1}}\right)dx=\boldsymbol{x\sqrt{x^2+1}+C}\quad (C\text{ は積分定数})\quad \cdots\cdots(\text{答})$$

である．

- 正直に攻めるとなかなか大変です．

(解 1) $x=\tan\theta$ とおく．

$$dx=\frac{1}{\cos^2\theta}\,d\theta$$

これより

$$\int\sqrt{x^2+1}\,dx=\int\sqrt{\tan^2\theta+1}\frac{1}{\cos^2\theta}\,d\theta=\int\frac{1}{\cos^3\theta}\,d\theta$$

$$\int\frac{1}{\sqrt{x^2+1}}\,dx=\int\frac{1}{\sqrt{\tan^2\theta+1}}\frac{1}{\cos^2\theta}\,d\theta=\int\frac{1}{\cos\theta}\,d\theta$$

ここで

$$\begin{aligned}\int\frac{1}{\cos^3\theta}\,d\theta &= \int\frac{1}{\cos\theta}(\tan\theta)'\,d\theta\\ &= \frac{1}{\cos\theta}\tan\theta-\int\frac{\sin\theta}{\cos^2\theta}\tan\theta\,d\theta\\ &= \frac{\sin\theta}{\cos^2\theta}-\int\frac{\sin^2\theta}{\cos^3\theta}\,d\theta\\ &= \frac{\sin\theta}{\cos^2\theta}-\int\left(\frac{1}{\cos^3\theta}-\frac{1}{\cos\theta}\right)d\theta\end{aligned}$$

$$\begin{aligned}\therefore\quad \int\left(\frac{2}{\cos^3\theta}-\frac{1}{\cos\theta}\right)d\theta &= \frac{\sin\theta}{\cos^2\theta}+C\quad (C\text{ は積分定数})\\ &= \tan\theta\cdot\frac{1}{\cos\theta}+C\end{aligned}$$

すなわち

$$\int\left(2\sqrt{x^2+1}-\frac{1}{\sqrt{x^2+1}}\right)dx=x\sqrt{x^2+1}+C$$

である．

(解 2) $t = x + \sqrt{x^2+1}$ とおく．

$$dt = \left(1 + \frac{2x}{2\sqrt{x^2+1}}\right)dx = \frac{\sqrt{x^2+1}+x}{\sqrt{x^2+1}}\,dx = \frac{t}{\sqrt{x^2+1}}dx$$

これより

$$\begin{aligned}\int \frac{1}{\sqrt{x^2+1}}\,dx &= \int \frac{dt}{t} = \log|t| + C \quad (C \text{ は積分定数})\\ &= \log(x+\sqrt{x^2+1}) + C\end{aligned}$$

である．

また $t - x = \sqrt{x^2+1}$ より

$$t^2 - 2tx + x^2 = x^2 + 1$$

$$\therefore \quad x = \frac{t^2-1}{2t} = \frac{1}{2}\left(t - \frac{1}{t}\right)$$

$$\therefore \quad dx = \frac{1}{2}\left(1 + \frac{1}{t^2}\right)dt$$

これより

$$\begin{aligned}\int \sqrt{x^2+1}\,dx &= \int (t-x)\cdot\frac{1}{2}\left(1+\frac{1}{t^2}\right)dt\\ &= \int \left\{t - \frac{1}{2}\left(t-\frac{1}{t}\right)\right\}\cdot\frac{1}{2}\left(1+\frac{1}{t^2}\right)dt\\ &= \frac{1}{4}\int\left(t+\frac{1}{t}\right)\left(1+\frac{1}{t^2}\right)dt\\ &= \frac{1}{4}\int\left(t+\frac{2}{t}+\frac{1}{t^3}\right)dt\\ &= \frac{1}{4}\left(\frac{t^2}{2}+2\log|t|-\frac{1}{2t^2}\right)+C\\ &= \frac{1}{8}\left(t+\frac{1}{t}\right)\left(t-\frac{1}{t}\right)+\frac{1}{2}\log|t|+C\end{aligned}$$

ここで，$t = x+\sqrt{x^2+1}$，$t - \dfrac{1}{t} = 2x$，$t + \dfrac{1}{t} = 2\sqrt{x^2+1}$ であるから

$$\begin{aligned}\int \sqrt{x^2+1}\,dx &= \frac{1}{8}\cdot 2\sqrt{x^2+1}\cdot 2x + \frac{1}{2}\log(x+\sqrt{x^2+1}) + C\\ &= \frac{1}{2}x\sqrt{x^2+1} + \frac{1}{2}\log(x+\sqrt{x^2+1}) + C\end{aligned}$$

である．

以上より

$$\begin{aligned}&\int\left(2\sqrt{x^2+1} - \frac{1}{\sqrt{x^2+1}}\right)dx\\ &= 2\left\{\frac{1}{2}x\sqrt{x^2+1} + \frac{1}{2}\log(x+\sqrt{x^2+1})\right\} - \log(x+\sqrt{x^2+1}) + C\\ &= x\sqrt{x^2+1} + C\end{aligned}$$

を得る．

(解 3) $x = \dfrac{e^t - e^{-t}}{2}$ とおくと

$$dx = \frac{e^t + e^{-t}}{2}\,dt$$

また

$$\sqrt{x^2+1} = \sqrt{\left(\frac{e^t - e^{-t}}{2}\right)^2 + 1} = \sqrt{\left(\frac{e^t + e^{-t}}{2}\right)^2} = \frac{e^t + e^{-t}}{2}$$

であるから

$$\begin{aligned}\int \sqrt{x^2+1}\,dx &= \int \frac{e^t+e^{-t}}{2}\cdot\frac{e^t+e^{-t}}{2}\,dt \\ &= \frac{1}{4}\int (e^{2t}+2+e^{-2t})\,dt \\ &= \frac{1}{4}\left(\frac{e^{2t}}{2}+2t-\frac{e^{-2t}}{2}\right)+C \quad (C\text{ は積分定数}) \\ &= \frac{(e^t-e^{-t})(e^t+e^{-t})}{8}+\frac{1}{2}t+C\end{aligned}$$

$$\int \frac{1}{\sqrt{x^2+1}}\,dx = \int \frac{2}{e^t+e^{-t}}\cdot\frac{e^t+e^{-t}}{2}\,dt = t+C$$

したがって

$$\begin{aligned}&\int \left(2\sqrt{x^2+1}-\frac{1}{\sqrt{x^2+1}}\right)dx \\ &= \left\{\frac{(e^t-e^{-t})(e^t+e^{-t})}{4}+t\right\}-t+C \\ &= \frac{(e^t-e^{-t})}{2}\,\frac{(e^t+e^{-t})}{2}+C \\ &= x\sqrt{x^2+1}+C\end{aligned}$$

である.