次の問に答えよ．

(1) 等式 $(\tan\theta)' = \dfrac{1}{\cos^2\theta}$ を示せ．また，定積分 $\displaystyle\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{1}{\sin^2\theta}\,d\theta$ の値を求めよ．

(2) 等式 $\dfrac{\cos\theta}{1+\sin\theta} + \dfrac{\cos\theta}{1-\sin\theta} = \dfrac{2}{\cos\theta}$ を示せ．また，定積分 $\displaystyle\int_0^{\frac{\pi}{6}} \frac{1}{\cos\theta}\,d\theta$ の値を求めよ．

(3) 定積分 $\displaystyle\int_0^{\frac{\pi}{6}} \frac{1}{\cos^3\theta}\,d\theta$ の値を求めよ

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【答】

(1) 等式の証明は略． $\displaystyle\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{1}{\sin^2\theta}\,d\theta = 1$

(2) 等式の証明は略． $\displaystyle\int_0^{\frac{\pi}{6}} \frac{1}{\cos\theta}\,d\theta = \frac{\log 3}{2}$

(3) $\displaystyle\int_0^{\frac{\pi}{6}} \frac{1}{\cos^3\theta}\,d\theta = \frac{1}{3} + \frac{\log 3}{4}$

【解答】

(1) $(\sin\theta)' = \cos\theta$, $(\cos\theta)' = -\sin\theta$ を認めて，商の微分を用いると

$$(\tan\theta)' = \left(\frac{\sin\theta}{\cos\theta}\right)' = \frac{(\sin\theta)'\cdot\cos\theta - \sin\theta\cdot(\cos\theta)'}{\cos^2\theta}$$

$$= \frac{\cos^2\theta + \sin^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta} \qquad \cdots\cdots(\text{証明終わり})$$

である．

次に $\displaystyle\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{1}{\sin^2\theta}\,d\theta$ を求める． $\theta = \dfrac{\pi}{2} - \varphi$ とおくと

$d\theta = -d\varphi$

θ	$\frac{\pi}{4}$	$\longrightarrow$	$\frac{\pi}{2}$
φ	$\frac{\pi}{4}$	$\longrightarrow$	0

であるから

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{1}{\sin^2\theta}\,d\theta = \int_{\frac{\pi}{4}}^{0} \frac{1}{\sin^2\left(\frac{\pi}{2}-\varphi\right)}(-1)\,d\varphi = \int_0^{\frac{\pi}{4}} \frac{1}{\cos^2\varphi}\,d\varphi$$

$\tan\varphi$ は $\dfrac{1}{\cos^2\varphi}$ の原始関数なので

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{1}{\sin^2\theta}\,d\theta = \int_0^{\frac{\pi}{4}} \frac{1}{\cos^2\varphi}\,d\varphi = \Big[\tan\varphi\Big]_0^{\frac{\pi}{4}} = \mathbf{1} \qquad \cdots\cdots(\text{答})$$

である．

(2) 左辺を整理していくと

$$\frac{\cos\theta}{1+\sin\theta} + \frac{\cos\theta}{1-\sin\theta} = \frac{\cos\theta\{(1-\sin\theta)+(1+\sin\theta)\}}{(1+\sin\theta)(1-\sin\theta)}$$

$$= \frac{2\cos\theta}{\cos^2\theta} = \frac{2}{\cos\theta} \qquad \cdots\cdots(\text{証明終わり})$$

となる．これにより

$$
\begin{aligned}
\int_0^{\frac{\pi}{6}} \frac{1}{\cos\theta}\,d\theta &= \int_0^{\frac{\pi}{6}} \frac{1}{2}\left(\frac{\cos\theta}{1+\sin\theta} + \frac{\cos\theta}{1-\sin\theta}\right)d\theta \\
&= \frac{1}{2}\int_0^{\frac{\pi}{6}} \left\{\frac{(1+\sin\theta)'}{1+\sin\theta} - \frac{(1-\sin\theta)'}{1-\sin\theta}\right\}d\theta \\
&= \frac{1}{2}\left(\Big[\log|1+\sin\theta|\Big]_0^{\frac{\pi}{6}} - \Big[\log|1-\sin\theta|\Big]_0^{\frac{\pi}{6}}\right) \\
&= \frac{1}{2}\left(\log\frac{3}{2} - \log\frac{1}{2}\right) \\
&= \frac{\mathbf{log\,3}}{\mathbf{2}} \qquad \cdots\cdots(\text{答})
\end{aligned}
$$

である．

(3) (1) の議論と部分積分法により

$$
\begin{aligned}
\int_0^{\frac{\pi}{6}} \frac{1}{\cos^3\theta}\,d\theta &= \int_0^{\frac{\pi}{6}} (\tan\theta)'\cdot\frac{1}{\cos\theta}\,d\theta \\
&= \left[\tan\theta\cdot\frac{1}{\cos\theta}\right]_0^{\frac{\pi}{6}} - \int_0^{\frac{\pi}{6}} \tan\theta\cdot\left(\frac{1}{\cos\theta}\right)'d\theta \\
&= \frac{1}{\sqrt{3}}\cdot\frac{1}{\frac{\sqrt{3}}{2}} - \int_0^{\frac{\pi}{6}} \tan\theta\cdot\frac{\sin\theta}{\cos^2\theta}\,d\theta \\
&= \frac{2}{3} - \int_0^{\frac{\pi}{6}} \frac{\sin^2\theta}{\cos^3\theta}\,d\theta \\
&= \frac{2}{3} - \int_0^{\frac{\pi}{6}} \frac{1-\cos^2\theta}{\cos^3\theta}\,d\theta \\
&= \frac{2}{3} - \int_0^{\frac{\pi}{6}} \frac{1}{\cos^3\theta}\,d\theta + \int_0^{\frac{\pi}{6}} \frac{1}{\cos\theta}\,d\theta
\end{aligned}
$$

となる．式を整理し直すと

$$
\begin{aligned}
2\int_0^{\frac{\pi}{6}} \frac{1}{\cos^3\theta}\,d\theta &= \frac{2}{3} + \int_0^{\frac{\pi}{6}} \frac{1}{\cos\theta}\,d\theta \\
&= \frac{2}{3} + \frac{\log 3}{2} \quad (\because\ (2))
\end{aligned}
$$

$$
\therefore\quad \int_0^{\frac{\pi}{6}} \frac{1}{\cos^3\theta}\,d\theta = \frac{\mathbf{1}}{\mathbf{3}} + \frac{\mathbf{log\,3}}{\mathbf{4}} \qquad \cdots\cdots(\text{答})
$$

である．