以下の問いに答えよ．

(1) 次の関数の導関数を求めよ．

(i) $\log|x+\sqrt{1+x^2}|$

(ii) $\dfrac{1}{2}(x\sqrt{1+x^2}+\log|x+\sqrt{1+x^2}|)$

(2) 次の等式を示せ．

$$\int_0^{\frac{\pi}{2}} \frac{\cos^3 x}{\sqrt{1+\sin^2 x}}\,dx = \frac{1}{2}\{3\log(1+\sqrt{2})-\sqrt{2}\}$$

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【答】

(1) (i) $\log|x+\sqrt{1+x^2}| = \dfrac{1}{\sqrt{1+x^2}}$

(ii) $\dfrac{1}{2}(x\sqrt{1+x^2}+\log|x+\sqrt{1+x^2}|) = \sqrt{1+x^2}$

(2) 略

【解答】

(1) (i) 合成関数の微分法を用いると

$$\begin{aligned}
&\left(\log|x+\sqrt{1+x^2}|\right)' \\
&= \frac{(x+\sqrt{1+x^2})'}{x+\sqrt{1+x^2}} = \frac{1+\frac{1}{2}(1+x^2)^{-\frac{1}{2}}\cdot(2x)}{x+\sqrt{1+x^2}} \\
&= \frac{1+\frac{x}{\sqrt{1+x^2}}}{x+\sqrt{1+x^2}} = \frac{\frac{\sqrt{1+x^2}+x}{\sqrt{1+x^2}}}{x+\sqrt{1+x^2}} \\
&= \boldsymbol{\frac{1}{\sqrt{1+x^2}}} \qquad \cdots\cdots(\text{答})
\end{aligned}$$

である．

(ii) 積の微分法，合成関数の微分法を用いると

$$\begin{aligned}
&\left(x\sqrt{1+x^2}\right)' \\
&= 1\cdot\sqrt{1+x^2}+x\cdot\frac{1}{2}(1+x^2)^{-\frac{1}{2}}\cdot(2x) \\
&= \sqrt{1+x^2}+\frac{x^2}{\sqrt{1+x^2}} \\
&= \frac{1+2x^2}{\sqrt{1+x^2}}
\end{aligned}$$

であるから，(i) の結果もあわせると

$$\begin{aligned}
&\frac{1}{2}\left(x\sqrt{1+x^2}+\log|x+\sqrt{1+x^2}|\right)' \\
&= \frac{1}{2}\left(x\sqrt{1+x^2}\right)'+\frac{1}{2}\left(\log|x+\sqrt{1+x^2}|\right)' \\
&= \frac{1}{2}\cdot\frac{1+2x^2}{\sqrt{1+x^2}}+\frac{1}{2}\cdot\frac{1}{\sqrt{1+x^2}} \\
&= \frac{2+2x^2}{2\sqrt{1+x^2}} \\
&= \boldsymbol{\sqrt{1+x^2}} \qquad \cdots\cdots(\text{答})
\end{aligned}$$

である．

(2) $$\int_0^{\frac{\pi}{2}} \frac{\cos^3 x}{\sqrt{1+\sin^2 x}}\,dx = \int_0^{\frac{\pi}{2}} \frac{1-\sin^2 x}{\sqrt{1+\sin^2 x}}\cos x\,dx$$

$u=\sin x$ とおくと

$$du = \cos x\,dx$$

x	0	$\longrightarrow$	$\frac{\pi}{2}$
u	0	$\longrightarrow$	1

であるから

$$\begin{aligned}
&\int_0^{\frac{\pi}{2}} \frac{\cos^3 x}{\sqrt{1+\sin^2 x}}\,dx \\
&= \int_0^1 \frac{1-u^2}{\sqrt{1+u^2}}\,du = \int_0^1 \frac{2-(1+u^2)}{\sqrt{1+u^2}}\,du \\
&= \int_0^1 \left(\frac{2}{\sqrt{1+u^2}} - \sqrt{1+u^2}\right) du \\
&= 2\Big[\log|u+\sqrt{1+u^2}|\Big]_0^1 - \frac{1}{2}\Big[u\sqrt{1+u^2} + \log|u+\sqrt{1+u^2}|\Big]_0^1 \\
&\qquad (\because\ (1)\text{ の }(\text{i})(\text{ii})) \\
&= \frac{3}{2}\Big[\log|u+\sqrt{1+u^2}|\Big]_0^1 - \frac{1}{2}\Big[u\sqrt{1+u^2}\Big]_0^1 \\
&= \frac{3}{2}\log(1+\sqrt{2}) - \frac{1}{2}\cdot\sqrt{2} \\
&= \frac{1}{2}\{3\log(1+\sqrt{2}) - \sqrt{2}\}
\end{aligned}$$

…… (証明終わり)

である.