定積分

$$\int_{\log\frac{\pi}{4}}^{\log\frac{\pi}{2}} \frac{e^{2x}}{(\sin(e^x))^2}\,dx$$

を求めよ．

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【答】 $\dfrac{\pi}{4}+\dfrac{1}{2}\log 2$

【解答】

$$I=\int_{\log\frac{\pi}{4}}^{\log\frac{\pi}{2}} \frac{e^{2x}}{(\sin(e^x))^2}\,dx$$

とおく．$e^x=t$ とおくと

$$e^x\,dx=dt \qquad \therefore\quad dx=\frac{1}{t}\,dt$$

x	$\log\frac{\pi}{4}$	$\longrightarrow$	$\log\frac{\pi}{2}$
t	$\frac{\pi}{4}$	$\longrightarrow$	$\frac{\pi}{2}$

であるから

$$I=\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{t^2}{\sin^2 t}\cdot\frac{1}{t}\,dt=\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{t}{\sin^2 t}\,dt$$

さらに，$t=\dfrac{\pi}{2}-u$ とおくと

$$dt=-du$$

t	$\frac{\pi}{4}$	$\longrightarrow$	$\frac{\pi}{2}$
u	$\frac{\pi}{4}$	$\longrightarrow$	0

であるから

$$\begin{aligned}
I&=\int_{\frac{\pi}{4}}^{0} \frac{\frac{\pi}{2}-u}{\sin^2\left(\frac{\pi}{2}-u\right)}(-1)\,du=\int_0^{\frac{\pi}{4}} \frac{\frac{\pi}{2}-u}{\cos^2 u}\,du\\
&=\left[\left(\frac{\pi}{2}-u\right)\tan u\right]_0^{\frac{\pi}{4}}-\int_0^{\frac{\pi}{4}}(-1)\tan u\,du\\
&=\frac{\pi}{4}-\int_0^{\frac{\pi}{4}}\frac{-\sin u}{\cos u}\,du\\
&=\frac{\pi}{4}-\Big[\log|\cos u|\Big]_0^{\frac{\pi}{4}}\\
&=\frac{\pi}{4}-\log\frac{1}{\sqrt{2}}\\
&=\boldsymbol{\frac{\pi}{4}+\frac{1}{2}\log 2} \qquad \cdots\cdots(\text{答})
\end{aligned}$$

である．

- $\left(\dfrac{\cos t}{\sin t}\right)'=\dfrac{-\sin t\cdot\sin t-\cos t\cdot\cos t}{\sin^2 t}=-\dfrac{1}{\sin^2 t}$ なので

$$\int\frac{1}{\sin^2 t}\,dt=-\frac{\cos t}{\sin t}+C \quad (C\text{ は積分定数})$$

であるから

$$
\begin{aligned}
I &= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{t}{\sin^2 t}\,dt \\
&= \left[t\left(-\frac{\cos t}{\sin t}\right)\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}} - \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} 1\cdot\left(-\frac{\cos t}{\sin t}\right)\,dt \\
&= \frac{\pi}{4} + \Big[\,\log|\sin t|\Big]_{\frac{\pi}{4}}^{\frac{\pi}{2}} \\
&= \frac{\pi}{4} - \log\frac{1}{\sqrt{2}} \\
&= \frac{\pi}{4} + \frac{1}{2}\log 2
\end{aligned}
$$

である.