以下の 3 つの定積分の値を求めよ.

$$\int_1^4 \frac{3+\sqrt{x}}{x^2}\,dx = \frac{\boxed{10}\boxed{11}}{\boxed{12}}$$

$$\int_0^\pi \sin^3 x\,dx = \frac{\boxed{13}}{\boxed{14}}$$

$$\int_0^\pi \sin^4 x\,dx = \frac{\boxed{15}}{\boxed{16}}\pi$$

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【答】

1011	12	13	14	15	16
13	4	4	3	3	8

【解答】

それぞれの値は

$$\begin{aligned}\int_1^4 \frac{3+\sqrt{x}}{x^2}\,dx &= \int_1^4 (3x^{-2}+x^{-\frac{3}{2}})\,dx \\ &= \Big[-3x^{-1}-2x^{-\frac{1}{2}}\Big]_1^4 \\ &= -3\left(\frac{1}{4}-1\right)-2\left(\frac{1}{2}-1\right) \\ &= \frac{9}{4}+1 \\ &= \boldsymbol{\frac{13}{4}}\end{aligned}$$

……(答)

$$\begin{aligned}\int_0^\pi \sin^3 x\,dx &= -\int_0^\pi (1-\cos^2 x)(\cos x)'\,dx \\ &= -\left[\cos x-\frac{1}{3}\cos^3 x\right]_0^\pi \\ &= -\left\{(-1-1)-\frac{1}{3}(-1-1)\right\} \\ &= 2-\frac{2}{3} \\ &= \boldsymbol{\frac{4}{3}}\end{aligned}$$

……(答)

最後に

$$\begin{aligned}\sin^4 x &= \left(\frac{1-\cos 2x}{2}\right)^2 \\ &= \frac{1-2\cos 2x+\cos^2 2x}{4} \\ &= \frac{1}{4}-\frac{1}{2}\cos 2x+\frac{1}{4}\cdot\frac{1+\cos 4x}{2} \\ &= \frac{3}{8}-\frac{1}{2}\cos 2x+\frac{1}{8}\cos 4x\end{aligned}$$

であるから

$$\begin{aligned}\int_0^\pi \sin^4 x\,dx &= \int_0^\pi \left(\frac{3}{8}-\frac{1}{2}\cos 2x+\frac{1}{8}\cos 4x\right)dx \\ &= \left[\frac{3}{8}x-\frac{1}{4}\sin 2x+\frac{1}{32}\sin 4x\right]_0^\pi \\ &= \boldsymbol{\frac{3}{8}\pi}\end{aligned}$$

……(答)

である.

- 3 倍角の公式 $\sin 3x = 3\sin x - 4\sin^3 x$ を用いると

$$\begin{aligned}\int_0^\pi \sin^3 x\,dx &= \frac{1}{4}\int_0^\pi (3\sin x - \sin 3x)\,dx \\ &= \frac{1}{4}\Big[-3\cos x + \frac{1}{3}\cos 3x\Big]_0^\pi \\ &= \frac{4}{3}\end{aligned}$$

- $I(n) = \displaystyle\int_0^\pi \sin^n x\,dx$ とおく．部分積分法を用いると

$$\begin{aligned}I(n) &= \int_0^\pi \sin^{n-1} x(-\cos x)'\,dx \\ &= -\Big[\sin^{n-1} x\cdot\cos x\Big]_0^\pi + \int_0^\pi (n-1)\sin^{n-2}\cos x\cdot\cos x\,dx \\ &= 0 + \int_0^\pi (n-1)\sin^{n-2}(1-\sin^2 x)\,dx \\ &= (n-1)\{I(n-2) - I(n)\}\end{aligned}$$

が成り立つ．式を整理すると

$$\begin{aligned}&\{1+(n-1)\}I(n) = (n-1)I(n-2) \\ \therefore\quad &I(n) = \frac{n-1}{n}I(n-2)\end{aligned}$$

を得る．よって

$$\begin{aligned}\int_0^\pi \sin^3 x\,dx &= \frac{2}{3}I(1) = \frac{2}{3}\Big[-\cos x\Big]_0^\pi = \frac{2}{3}\cdot(1+1) = \frac{4}{3} \\ \int_0^\pi \sin^4 x\,dx &= \frac{3}{4}\cdot\frac{1}{2}I(0) = \frac{3}{4}\cdot\frac{1}{2}\Big[x\Big]_0^\pi = \frac{3}{4}\cdot\frac{1}{2}\cdot\pi = \frac{3}{8}\pi\end{aligned}$$

である．