1 辺の長さが 2 の正三角形とその内接円の接点を A，B，C とする．点 P が内接円の円周上にあるとき，以下の設問に答えよ．

(1) 内接円の中心を O とするとき，線分 OA の長さを求めよ．
(2) $\overrightarrow{\mathrm{PA}}\cdot\overrightarrow{\mathrm{PB}}+\overrightarrow{\mathrm{PB}}\cdot\overrightarrow{\mathrm{PC}}+\overrightarrow{\mathrm{PC}}\cdot\overrightarrow{\mathrm{PA}}$ の値を求めよ．
(3) $|\overrightarrow{\mathrm{PA}}|^2+|\overrightarrow{\mathrm{PB}}|^2+|\overrightarrow{\mathrm{PC}}|^2$ の値を求めよ．
(4) 点 P が円周上を動くとき，$\overrightarrow{\mathrm{PA}}\cdot\overrightarrow{\mathrm{PB}}$ の最大値および最小値を求めよ．

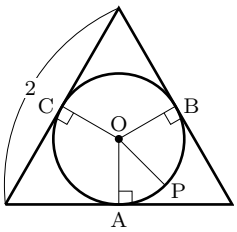
(23 関西医大 3)

【答】

(1) $\mathrm{OA}=\dfrac{\sqrt{3}}{3}$
(2) $\dfrac{1}{2}$
(3) 2
(4) 最大値 $\dfrac{1}{2}$，最小値 $-\dfrac{1}{6}$

【解答】

(1) 正三角形の内心 O は重心でもあるから，OA は一つの頂点から対辺に下ろした垂線の長さの $\dfrac{1}{3}$ である．



$$\mathrm{OA}=\frac{1}{3}\times 2\sin\frac{\pi}{3}=\boldsymbol{\frac{\sqrt{3}}{3}} \qquad \cdots\cdots(\text{答})$$

である．

(2) 始点を O にそろえると

$$\begin{aligned}
&\overrightarrow{\mathrm{PA}}\cdot\overrightarrow{\mathrm{PB}}+\overrightarrow{\mathrm{PB}}\cdot\overrightarrow{\mathrm{PC}}+\overrightarrow{\mathrm{PC}}\cdot\overrightarrow{\mathrm{PA}}\\
&=(\overrightarrow{\mathrm{OA}}-\overrightarrow{\mathrm{OP}})\cdot(\overrightarrow{\mathrm{OB}}-\overrightarrow{\mathrm{OP}})+(\overrightarrow{\mathrm{OB}}-\overrightarrow{\mathrm{OP}})\cdot(\overrightarrow{\mathrm{OC}}-\overrightarrow{\mathrm{OP}})+(\overrightarrow{\mathrm{OC}}-\overrightarrow{\mathrm{OP}})\cdot(\overrightarrow{\mathrm{OA}}-\overrightarrow{\mathrm{OP}})\\
&=\{\overrightarrow{\mathrm{OA}}\cdot\overrightarrow{\mathrm{OB}}-(\overrightarrow{\mathrm{OA}}+\overrightarrow{\mathrm{OB}})\cdot\overrightarrow{\mathrm{OP}}+|\overrightarrow{\mathrm{OP}}|^2\}+\{\overrightarrow{\mathrm{OB}}\cdot\overrightarrow{\mathrm{OC}}-(\overrightarrow{\mathrm{OB}}+\overrightarrow{\mathrm{OC}})\cdot\overrightarrow{\mathrm{OP}}+|\overrightarrow{\mathrm{OP}}|^2\}\\
&\quad+\{\overrightarrow{\mathrm{OC}}\cdot\overrightarrow{\mathrm{OA}}-(\overrightarrow{\mathrm{OC}}+\overrightarrow{\mathrm{OA}})\cdot\overrightarrow{\mathrm{OP}}+|\overrightarrow{\mathrm{OP}}|^2\}\\
&=(\overrightarrow{\mathrm{OA}}\cdot\overrightarrow{\mathrm{OB}}+\overrightarrow{\mathrm{OB}}\cdot\overrightarrow{\mathrm{OC}}+\overrightarrow{\mathrm{OC}}\cdot\overrightarrow{\mathrm{OA}})-2(\overrightarrow{\mathrm{OA}}+\overrightarrow{\mathrm{OB}}+\overrightarrow{\mathrm{OC}})\cdot\overrightarrow{\mathrm{OP}}+3|\overrightarrow{\mathrm{OP}}|^2
\end{aligned}$$

$|\overrightarrow{\mathrm{OA}}|=|\overrightarrow{\mathrm{OB}}|=|\overrightarrow{\mathrm{OC}}|=|\overrightarrow{\mathrm{OP}}|=\dfrac{\sqrt{3}}{3}$，$\angle\mathrm{AOB}=\angle\mathrm{BOC}=\angle\mathrm{COA}=\dfrac{2\pi}{3}$ より，$\overrightarrow{\mathrm{OA}}+\overrightarrow{\mathrm{OB}}+\overrightarrow{\mathrm{OC}}=\vec{0}$ でもあるから

$$\begin{aligned}
&\overrightarrow{\mathrm{PA}}\cdot\overrightarrow{\mathrm{PB}}+\overrightarrow{\mathrm{PB}}\cdot\overrightarrow{\mathrm{PC}}+\overrightarrow{\mathrm{PC}}\cdot\overrightarrow{\mathrm{PA}}\\
&=3\times\left(\frac{\sqrt{3}}{3}\right)^2\cos\frac{2\pi}{3}-0+3\left(\frac{\sqrt{3}}{3}\right)^2\\
&=-\frac{1}{2}-0+1\\
&=\boldsymbol{\frac{1}{2}} \qquad \cdots\cdots(\text{答})
\end{aligned}$$

である．

(3) 始点を O にそろえると

$$\begin{aligned}
&|\overrightarrow{\mathrm{PA}}|^2+|\overrightarrow{\mathrm{PB}}|^2+|\overrightarrow{\mathrm{PC}}|^2\\
&=|\overrightarrow{\mathrm{OA}}-\overrightarrow{\mathrm{OP}}|^2+|\overrightarrow{\mathrm{OB}}-\overrightarrow{\mathrm{OP}}|^2+|\overrightarrow{\mathrm{OC}}-\overrightarrow{\mathrm{OP}}|^2\\
&=\left\{|\overrightarrow{\mathrm{OA}}|^2-2\overrightarrow{\mathrm{OA}}\cdot\overrightarrow{\mathrm{OP}}+|\overrightarrow{\mathrm{OP}}|^2\right\}+\left\{|\overrightarrow{\mathrm{OB}}|^2-2\overrightarrow{\mathrm{OB}}\cdot\overrightarrow{\mathrm{OP}}+|\overrightarrow{\mathrm{OP}}|^2\right\}\\
&\quad+\left\{|\overrightarrow{\mathrm{OC}}|^2-2\overrightarrow{\mathrm{OC}}\cdot\overrightarrow{\mathrm{OP}}+|\overrightarrow{\mathrm{OP}}|^2\right\}\\
&=(|\overrightarrow{\mathrm{OA}}|^2+|\overrightarrow{\mathrm{OB}}|^2+|\overrightarrow{\mathrm{OC}}|^2)-2(\overrightarrow{\mathrm{OA}}+\overrightarrow{\mathrm{OB}}+\overrightarrow{\mathrm{OC}})\cdot\overrightarrow{\mathrm{OP}}+3|\overrightarrow{\mathrm{OP}}|^2\\
&=3\times\left(\frac{\sqrt{3}}{3}\right)^2-0+3\times\left(\frac{\sqrt{3}}{3}\right)^2\\
&=1-0+1\\
&=\mathbf{2}
\end{aligned}$$

……(答)

である.

(4) 始点を O にそろえると

$$\begin{aligned}
\overrightarrow{\mathrm{PA}}\cdot\overrightarrow{\mathrm{PB}}&=(\overrightarrow{\mathrm{OA}}-\overrightarrow{\mathrm{OP}})\cdot(\overrightarrow{\mathrm{OB}}-\overrightarrow{\mathrm{OP}})\\
&=\overrightarrow{\mathrm{OA}}\cdot\overrightarrow{\mathrm{OB}}-(\overrightarrow{\mathrm{OA}}+\overrightarrow{\mathrm{OB}})\cdot\overrightarrow{\mathrm{OP}}+|\overrightarrow{\mathrm{OP}}|^2\\
&=\left(\frac{\sqrt{3}}{3}\right)^2\cos\frac{2\pi}{3}+\overrightarrow{\mathrm{OC}}\cdot\overrightarrow{\mathrm{OP}}+\left(\frac{\sqrt{3}}{3}\right)^2\\
&=-\frac{1}{6}+\left(\frac{\sqrt{3}}{3}\right)^2\cos\angle\mathrm{COP}+\frac{1}{3}\\
&=\frac{1}{6}+\frac{1}{3}\cos\angle\mathrm{COP}
\end{aligned}$$

よって，$\overrightarrow{\mathrm{PA}}\cdot\overrightarrow{\mathrm{PB}}$ は

$\angle\mathrm{COP}=0$ のとき，最大値 $\dfrac{1}{6}+\dfrac{1}{3}\times1=\mathbf{\dfrac{1}{2}}$, ……(答)

$\angle\mathrm{COP}=\pi$のとき，最小値 $\dfrac{1}{6}+\dfrac{1}{3}\times(-1)=\mathbf{-\dfrac{1}{6}}$ ……(答)

をとる.