$n$ を自然数とし，$I_n = \displaystyle\int_0^{\frac{\pi}{3}} \frac{dx}{\cos^n x}$ とする．次の問いに答えよ．

(1) $I_1$，$I_2$ を求めよ．

(2) $n \geqq 2$ のとき，$I_{n+1}$ を $I_{n-1}$ を用いて表せ．

(3) $I_3$，$I_4$ を求めよ．

(4) 定積分 $\displaystyle\int_0^{\frac{\sqrt{3}}{2}} x^2\sqrt{1+4x^2}\,dx$ を求めよ．

(25 富山大 理・医・薬 2)

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【答】

(1) $I_1 = \log(2+\sqrt{3})$，$I_2 = \sqrt{3}$

(2) $I_{n+1} = \dfrac{2^{n-1}\sqrt{3}}{n} + \dfrac{n-1}{n} I_{n-1}$

(3) $I_3 = \sqrt{3} + \dfrac{1}{2}\log(2+\sqrt{3})$，$I_4 = 2\sqrt{3}$

(4) $\displaystyle\int_0^{\frac{\sqrt{3}}{2}} x^2\sqrt{1+4x^2}\,dx = \frac{7\sqrt{3}}{32} - \frac{1}{64}\log(2+\sqrt{3})$

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【解答】

$$I_n = \int_0^{\frac{\pi}{3}} \frac{dx}{\cos^n x} \quad (n \text{ は自然数})$$

(1) $I_1$ について

$$\begin{aligned}\frac{1}{\cos x} &= \frac{\cos x}{\cos^2 x} = \frac{\cos x}{(1+\sin x)(1-\sin x)} \\ &= \frac{1}{2}\left(\frac{\cos x}{1+\sin x} + \frac{\cos x}{1-\sin x}\right)\end{aligned}$$

となる．これにより

$$\begin{aligned}I_1 &= \int_0^{\frac{\pi}{3}} \frac{dx}{\cos x} \\ &= \frac{1}{2}\int_0^{\frac{\pi}{3}} \left\{\frac{(1+\sin x)'}{1+\sin x} - \frac{(1-\sin x)'}{1-\sin x}\right\} dx \\ &= \frac{1}{2}\left(\Big[\log|1+\sin x|\Big]_0^{\frac{\pi}{3}} - \Big[\log|1-\sin x|\Big]_0^{\frac{\pi}{3}}\right) \\ &= \frac{1}{2}\left(\log\frac{2+\sqrt{3}}{2} - \log\frac{2-\sqrt{3}}{2}\right) \\ &= \frac{1}{2}\log\frac{2+\sqrt{3}}{2-\sqrt{3}} \\ &= \frac{1}{2}\log(2+\sqrt{3})^2 \\ &= \boldsymbol{\log(2+\sqrt{3})}\end{aligned}$$

……(答)

である．

$I_2$ について

$$I_2 = \int_0^{\frac{\pi}{3}} \frac{dx}{\cos^2 x} = \Big[\tan x\Big]_0^{\frac{\pi}{3}} = \boldsymbol{\sqrt{3}}$$

……(答)

である．

(2) $n \geqq 2$ のとき

$$
\begin{aligned}
I_{n+1} &= \int_0^{\frac{\pi}{3}} \frac{dx}{\cos^{n+1} x} = \int_0^{\frac{\pi}{3}} \frac{(\tan x)'}{\cos^{n-1} x}\, dx \\
&= \left[ \frac{\tan x}{\cos^{n-1} x} \right]_0^{\frac{\pi}{3}} + (n-1) \int_0^{\frac{\pi}{3}} \frac{(\tan x)\cdot(-\sin x)}{\cos^n x}\, dx \\
&= \frac{\sqrt{3}}{\left(\frac{1}{2}\right)^{n-1}} - (n-1) \int_0^{\frac{\pi}{3}} \frac{\sin^2 x}{\cos^{n+1} x}\, dx \\
&= 2^{n-1}\sqrt{3} - (n-1) \int_0^{\frac{\pi}{3}} \frac{1-\cos^2 x}{\cos^{n+1} x}\, dx \\
&= 2^{n-1}\sqrt{3} - (n-1)(I_{n+1} - I_{n-1})
\end{aligned}
$$

$I_{n+1}$ をまとめると

$$\{1 + (n-1)\} I_{n+1} = 2^{n-1}\sqrt{3} + (n-1) I_{n-1}$$

$$\therefore \quad \boldsymbol{I_{n+1} = \frac{2^{n-1}\sqrt{3}}{n} + \frac{n-1}{n} I_{n-1}} \quad (n \geqq 2) \qquad \cdots\cdots(\text{答})$$

である．

(3) (1)，(2) より

$$I_3 = \frac{2^1\sqrt{3}}{2} + \frac{1}{2} I_1 = \boldsymbol{\sqrt{3} + \frac{1}{2}\log(2+\sqrt{3})} \qquad \cdots\cdots(\text{答})$$

$$I_4 = \frac{2^2\sqrt{3}}{3} + \frac{2}{3} I_2 = \frac{4\sqrt{3}}{3} + \frac{2}{3}\cdot\sqrt{3} = \boldsymbol{2\sqrt{3}} \qquad \cdots\cdots(\text{答})$$

である．

(4) $J = \displaystyle\int_0^{\frac{\sqrt{3}}{2}} x^2\sqrt{1+4x^2}\, dx$ とおく．$2x = \tan\theta$ とおくと

$$2\, dx = \frac{1}{\cos^2\theta}\, d\theta$$

| $x$ | $0$ | $\longrightarrow$ | $\frac{\sqrt{3}}{2}$ |
|---|---|---|---|
| $\theta$ | $0$ | $\longrightarrow$ | $\frac{\pi}{3}$ |

であるから

$$
\begin{aligned}
J &= \int_0^{\frac{\pi}{3}} \frac{\tan^2\theta}{4}\sqrt{1+\tan^2\theta}\cdot\frac{1}{2\cos^2\theta}\, d\theta \\
&= \frac{1}{8}\int_0^{\frac{\pi}{3}} \frac{\sin^2\theta}{\cos^5\theta}\, d\theta \\
&= \frac{1}{8}\int_0^{\frac{\pi}{3}} \frac{1-\cos^2\theta}{\cos^5\theta}\, d\theta \\
&= \frac{1}{8}(I_5 - I_3)
\end{aligned}
$$

ここで

$$
\begin{aligned}
I_5 - I_3 &= \left(\frac{2^3\sqrt{3}}{4} + \frac{3}{4} I_3\right) - I_3 = 2\sqrt{3} - \frac{1}{4} I_3 \\
&= 2\sqrt{3} - \frac{1}{4}\left\{\sqrt{3} + \frac{1}{2}\log(2+\sqrt{3})\right\} \\
&= \frac{7\sqrt{3}}{4} - \frac{1}{8}\log(2+\sqrt{3})
\end{aligned}
$$

であるから

$$
\begin{aligned}
J &= \frac{1}{8}\left\{\frac{7\sqrt{3}}{4} - \frac{1}{8}\log(2+\sqrt{3})\right\} \\
&= \boldsymbol{\frac{7\sqrt{3}}{32} - \frac{1}{64}\log(2+\sqrt{3})} \qquad \cdots\cdots(\text{答})
\end{aligned}
$$

である．